Super Beetle Gamer: An Introduction

Muhammad Adib Surani
adib.surani@gmail.com
The University of Melbourne

Cayden Liao
caydenliao1230@gmail.com
Carnegie Mellon University

24 August 2024

Abstract

In this short note we introduce the concept of a super beetle gamer, which is the minimal vector subspace of the infinite polynomial ring $\mathbb{Q}[x_1, x_2, ...]$ containing all polynomials that are determinants of some Vandermonde-like matrices. We also describe and prove a number of properties that are significant in this space.

1 Introduction

Let $\mathbb{Q}[x_1, x_2, ...]$ be the polynomial ring in the infinite set of variables $(x_1, x_2, ...)$ over $\mathbb{Q}$.

Definition 1.1: A (k, d)-*beetleshell* is a $k \times k$ matrix in which every row has the form

$$\left(x_i, ix_i, ..., i^{d-1}x_i, 1, i, ..., i^{k-d-1}\right)$$

over k distinct positive integers i.

Example: The following matrix is a (4,3)-beetleshell:

$$\begin{pmatrix} x_1 & x_1 & x_1 & 1 \\ x_7 & 7x_7 & 49x_7 & 1 \\ x_2 & 2x_2 & 4x_2 & 1 \\ x_9 & 9x_9 & 81x_9 & 1 \end{pmatrix}.$$

Remark: A $(k, 0)$-beetleshell is just a square Vandermonde matrix. Likewise, a (k, k)-beetleshell is a square Vandermonde matrix with every row multiplied by some x_i.

Definition 1.2: A polynomial is a (k, d)-*beetle* (or simply *beetle*) if it is the determinant of some (k, d)-beetleshell. It is also a *super beetle* if it is a (k, d)-beetle for some $k \geq 2d - 2$.

Example: The polynomial

$$-30x_1x_2x_7 + 56x_1x_2x_9 - 96x_1x_7x_9 + 70x_2x_7x_9$$

is a super beetle, as it is the determinant of the (4,3)-beetleshell from the previous example.

It should be immediately clear that a (k, d)-beetle must be a polynomial of degree d, and in fact each of its monomials has degree exactly d.

We have not yet motivated the condition $k \geq 2d - 2$ for a super beetle; it is difficult to do so directly, and we will instead define a space for these objects and see how this property is used.

Definition 1.3: The *super beetle gamer*[1] is the span of all super beetles (when treating $\mathbb{Q}[x_1, x_2, ...]$ as a vector space over $\mathbb{Q}$), and is denoted by SBG.

Theorem 1.1: SBG is closed under differentiation.

Proof: Since differentiation is linear, it suffices to prove that the derivative of any super beetle with respect to any variable x_i is also in SBG.

We first notice that SBG contains all constants (since 1 is a $(1, 0)$-beetle), as well as all affine maps (since x_i is a $(1, 1)$-beetle). This means that the derivative of any $(k, 0)$-beetle or $(k, 1)$-beetle must be in SBG.

So now let f be a super (k, d)-beetle with $d \geq 2$, and M its corresponding beetleshell. The case $d \leq 1$ is trivial, so assume $d \geq 2$. Partial differentiation of f on a single variable is equivalent to differentiating its corresponding row in M. (We assume the row exists, otherwise it just differentiates to zero.) Without loss of generality let this be the first row of M. This means that after differentiation, the first row of M' becomes

$$\left(1, i, ..., i^{d-1}, 0, 0, ..., 0\right).$$

The plan now is to perform some elementary row and column operations to show that the determinant of M' is in SBG. First, we subtract every column (multiplied by i) except columns k and d from the column on its immediate right. So this particular row becomes

$$(1, 0, ..., 0, 0, 0, ..., 0),$$

while a different row that was originally

$$\left(x_j, jx_j, ..., j^{d-1}x_j, 1, j, ..., j^{k-d-1}\right)$$

now becomes

$$\left(x_j, (j-i)x_j, ..., j^{d-2}(j-i)x_j, 1, j-i, ..., j^{k-d-2}(j-i)\right).$$

Recall that the determinant has not changed since these are just pivot operations. Since the first row only consists of a single 1 followed by all 0s, it suffices to only calculate the determinant of the bottom-right $(k-1) \times (k-1)$ matrix. This looks almost like a $(k-1, d-1)$-beetleshell where every row has been multiplied by some row-dependent constant $(j-i)$, except for a single unaffected column of 1s.

We can exploit this column of 1s by performing cofactor expansion down this colum, which means that our determinant becomes a linear combination of $(k-2, d-1)$-beetles. It only remains to check that these are in fact super beetles, and this is easily confirmed since $k - 2 \geq 2(d-1) - 2$. □

Remark: This also implies that SBG is closed under pre-composition with a translation, that is, if $f(x_1, x_2, ...) \in$ SBG, then $f(x_1 + \varepsilon_1, x_2 + \varepsilon_2, ...) \in$ SBG for every $\varepsilon_i \in \mathbb{Q}$.

[1] Named as such because gamers often strive to complete full collections efficiently.

At this point we would like to study some restrictions of SBG, in particular some finite-dimensional ones, as these can be used in practical settings.

Definition 1.4: The *super beetle gamer on n variables* is the restriction of SBG to $\mathbb{Q}[x_1, x_2, ..., x_n]$, and is denoted by SBG_n.

Definition 1.5: The *super beetle gamer of degree exactly d* is the restriction of SBG to polynomials where every monomial has degree exactly d, and is denoted by SBG^d. We can also combine with the previous definition to get SBG_n^d defined in the natural way.

Remark: It should be clear these are all still (vector) subspaces of SBG.

Theorem 1.2: Let n be a positive integer, and d a non-negative integer. Then

$$\dim\left(\mathrm{SBG}_n^d\right) = \begin{cases} 1 & \text{if } d = 0 \\ n & \text{if } d = 1 \\ \binom{n-d+2}{d} & \text{if } 2 \leq d \leq \frac{n+2}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The statement is obvious when $d \leq 2$, so suppose $d \geq 3$. For SBG_n^d to be non-trivial, it must contain at least one super (k, d)-beetle. Thus $2d - 2 \leq k \leq n$.

So now we assume $3 \leq d \leq \frac{n+2}{2}$. We will prove this by constructing an explicit basis of the required size.

Let (S, T) be a partition of the first n integers into sets of cardinality $n - d + 2$ and $d - 2$ respectively, that is, $S = \{1, 2, ..., n - d + 2\}$ and $T = \{n - d + 3, n - d + 4, ..., n\}$. Then let

$$X = \{S' \cup T : S' \subset S \text{ and } |S| = d\}$$

be a set whose elements are itself sets, all of cardinality $2d - 2$. Denote by $\zeta : X \to \mathrm{SBG}_n^d$ the canonical construction map, that is, given an $x \in X$ it constructs a $(2d - 2, d)$-beetle by taking the elements of x in ascending order to form the rows of the beetleshell. Then

$$B := \{\zeta(x) : x \in X\}$$

is a set B of super beetles with $|B| = \binom{n-d+2}{d}$. We now need to show that this is in fact a basis for SBG_n^d.

Linear independence is easy, since each subset S' is used exactly once, so its corresponding monomial appears in exactly one element of B.

To show that B spans SBG_n^d, we need to show that every super beetle can be produced from a linear combination of elements in B. Let f be a super beetle in SBG_n^d, and M its corresponding (k, d)-beetleshell.

The first step is to embed M into a larger (n, d)-beetleshell M' that has had its righmost $n - k$ columns zeroed out, except for a single 1 for each variable unused in f. The two matrices M and M' must then have the same determinant, possibly up to sign.

So M' has d columns with variables, followed by $k-d$ columns with constants, and finally followed by $n-k$ columns with one-hot encodings. Now because we want to for a linear combination of $(2d-2,d)$-beetles, we only care about the first $d-2$ columns of constants rather than the entire $k-d$. (We verify that is well-defined since $d-2 \leq k-d$.)

So the new perspective now is that M' has d columns with variables, followed by $d-2$ columns with constants, and finally followed by $n-2d+2$ columns of arbitrary constants. Now, the crucial step here is to pivot these rightmost $n-2d+2$ columns using the $d-2$ constant ones. We can always column-reduce this in such a way that the bottom-right $(d-2)\times(n-2d+2)$ submatrix is filled with zeroes.

Then we can evaluate the determinant of this "reduced" matrix by taking the cofactor expansion down these rightmost $n-2d+2$ columns. Since every element in the bottom-right $(d-2)\times(n-2d+2)$ submatrix is zero, it follows that every cofactor must contain the bottom $d-2$ variables.

It only remains to show that each cofactor here is in B (up to sign). This is easy to check since each cofactor is a $(2d-2,d)$-beetle by construction, and it must also contain the last $d-2$ variables.

Thus it follows that f is a linear combination of elements in B, as required.

Example: If $n=6$ and M is the $(4,3)$-beetleshell given by

$$M = \begin{pmatrix} x_1 & x_1 & x_1 & 1 \\ x_2 & 2x_2 & 4x_2 & 1 \\ x_3 & 3x_3 & 9x_3 & 1 \\ x_5 & 5x_5 & 25x_5 & 1 \end{pmatrix},$$

then

$$M' = \begin{pmatrix} x_1 & x_1 & x_1 & 1 & 0 & 0 \\ x_2 & 2x_2 & 4x_2 & 1 & 0 & 0 \\ x_3 & 3x_3 & 9x_3 & 1 & 0 & 0 \\ x_4 & 4x_4 & 16x_4 & 1 & 1 & 0 \\ x_5 & 5x_5 & 25x_5 & 1 & 0 & 0 \\ x_6 & 6x_6 & 36x_6 & 1 & 0 & 1 \end{pmatrix},$$

and we can column-reduce this to

$$M'' = \begin{pmatrix} x_1 & x_1 & x_1 & 1 & 0 & -1 \\ x_2 & 2x_2 & 4x_2 & 1 & 0 & -1 \\ x_3 & 3x_3 & 9x_3 & 1 & 0 & -1 \\ x_4 & 4x_4 & 16x_4 & 1 & 1 & -1 \\ x_5 & 5x_5 & 25x_5 & 1 & 0 & -1 \\ x_6 & 6x_6 & 36x_6 & 1 & 0 & 0 \end{pmatrix}$$

which is a matrix whose bottom-right 1×2 submatrix is filled with zeroes. Consequently, $\det(M'')$ is a linear combination of $(4,3)$-beetles that all contain x_6.

$\square$

Theorem 1.3: $\dim(\mathrm{SBG}_n) = F_{n+3} - 1$, where F_k is the kth Fibonacci number.

(Here we use the standard indexing of Fibonacci numbers whereby $F_0 = 0$ and $F_1 = 1$.)

Proof: This is a straightforward application of the well-known identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F_{n+1}$$

and the fact that SBG_n decomposes into the direct sum of its SBG_n^d components. □